THE BELL SYSTEM TECHNICAL JOURNAL
Vol. 51, No. 2, February, 1972
Printed in U.S.A.

Power Distribution and Radiation Losses in Multimode Dielectric Slab Waveguides

By D. MARCUSE

(Manuscript received September 1, 1971)

Guided modes of multimode waveguides exchange power if the waveguide deviates in any way from its perfect geometry. The power exchange problem is studied for a multimode slab waveguide under the assumption that the power coupling is caused by irregularities of the core-cladding interfaces. The problem is treated by means of coupled power equations. The main result of this study is the realization that the power distribution versus mode number settles down to a steady state distribution if the waveguide is sufficiently long. The shape of the steady state distribution depends on the correlation length of the function describing the core-cladding interface irregularities. For very short correlation length only the lowest-order mode carries an appreciable amount of power while the power carried by all the other modes is orders of magnitude smaller. For very long correlation length, on the other hand, all guided modes carry equal amounts of power. The steady state distribution is achieved regardless of the way in which the power was distributed over all the modes at the beginning of the guide. However, the total power in the steady state mode distribution is dependent on the initial power distribution.

I. INTRODUCTION

Light communications systems using optical fibers as the guidance medium are presently being planned for two different modes of operation. High-capacity systems are likely to be used with a laser as the light source and should be operated in the fundamental HE_{11} mode in order to minimize delay distortion that accompanies multimode operation. For less ambitious, low-capacity systems excitation of the fiber with a light emitting diode appears more economical. However, the output of light emitting diodes cannot be used to excite a single fiber mode with high efficiency. A low-capacity fiber to be used with a light emitting diode must thus be designed to operate with many modes.

Multimode optical fibers are not as easily characterized as single-mode

fibers. The power loss of such a fiber is usually not simply expressed by an exponential decay law but depends in a complicated way on the distribution of the power over the many modes. The present study is an attempt to describe the loss behavior of multimode optical fibers. We use the TE modes of the simplified model of a slab waveguide with the added requirement that there is no field variation or change in the slab geometry in the y direction of the coordinate system. This model makes it possible to describe the multimode waveguide rather simply. Even though it cannot directly be used to predict the loss behavior of round multimode optical fibers, it provides insight into the operating principles of multimode waveguides that can be used to obtain an understanding of the properties of multimode fibers of different shape. Our treatment of the multimode dielectric slab waveguide is based on coupled power equations. It has been shown in an earlier paper[1] that the coupled wave equations of a multimode optical waveguide[2] can be used to derive much simpler coupled power equations provided that the coupling mechanism can be described by a stationary random process with Gaussian correlation function. The coupled power equations have the advantage that their coefficient matrix is constant, real, and symmetric. The system of coupled linear first-order differential equations can thus be solved by first finding eigensolutions with the common z dependence $\exp(-\alpha z)$. These can be used to express the general solution as a superposition of eigensolutions. This approach makes it clear that a steady state power distribution must exist. By allowing the field to travel far enough in the waveguide, so that all but the lowest-loss eigensolution has decayed to insignificant values, it is obvious that the distribution of power over the many modes assumes the shape of the lowest-order eigensolution regardless of the initial power distribution. The power loss of the steady state eigensolution obeys a simple exponential law and can thus be characterized by a single number, the lowest-order eigenvalue of the eigensolutions of the power rate equations.

The mechanism causing coupling between the many guided modes and of guided modes to the continuous spectrum of radiation modes will be assumed to consist of irregularities of the core-cladding interface. The coupling coefficients for this model have been evaluated in an earlier paper.[2] Any imperfection of the refractive index distribution and the slab geometry causes coupling between the modes. We choose the core-cladding interface irregularities because this coupling mechanism is of fundamental importance and because its properties are well understood. Mode coupling caused by irregularities of the refractive

index distribution will cause similar effects in some respects. However, there are differences that consist mostly in the dependence of the coupling process on the mode number of the coupled modes.

The coupling coefficient for the core-cladding irregularities can be expressed as a product of a term that is independent of the length coordinate z but depends on the mode number times a z-dependent function that describes the actual shape of the core-cladding interface. This function $f(z)$ is assumed to be a stationary random variable with a Gaussian correlation function that can be completely described by the rms deviation $\bar{\sigma}$ of the core-cladding interface from a perfect plane and by the correlation length D. The same process that couples the guided modes among each other also causes each mode to lose power to the continuous spectrum of radiation modes. The interplay between coupling among the guided modes and power loss to radiation is responsible for the shape of the steady state distribution as well as for the loss associated with that steady state distribution.

In order to spare readers not interested in the details of the theory the trouble of finding their way through the theoretical part of the paper, we present the results of the numerical analysis before the discussion of the details of the theory.

II. RESULTS OF THE NUMERICAL ANALYSIS

The theory has been evaluated for a slab waveguide with a core index of $n_1 = 1.5$ and a core-to-cladding-index ratio of $n_1/n_2 = 1.01$. Most numerical results hold for a slab waveguide supporting ten modes corresponding to the value $kd = 82$ (k = free-space propagation constant, d = slab half width). The only other case for which numerical values have been calculated corresponds to $kd = 165$ with twenty-one guided modes. It has been assumed throughout that the irregularities of the two core-cladding interfaces are statistically independent of each other but have the same rms deviation and the same correlation length.

Figure 1 is a plot of the steady state distribution of the ten-mode slab waveguide. The steady state mode power is plotted versus mode number. Actually, only integer values of the mode number have physical meaning. In order to be able to display the mode power distributions for several values of the correlation length on one graph, the power values at the integer mode numbers were connected by straight lines. The label $B_n^{(1)}$ of the vertical axis refers to the lowest-order eigenvector of the eigenvalue problem [see equation (62) of the theoretical part]. These values are proportional to the power in each mode. They are normalized so that the squares of the power values for all ten integer

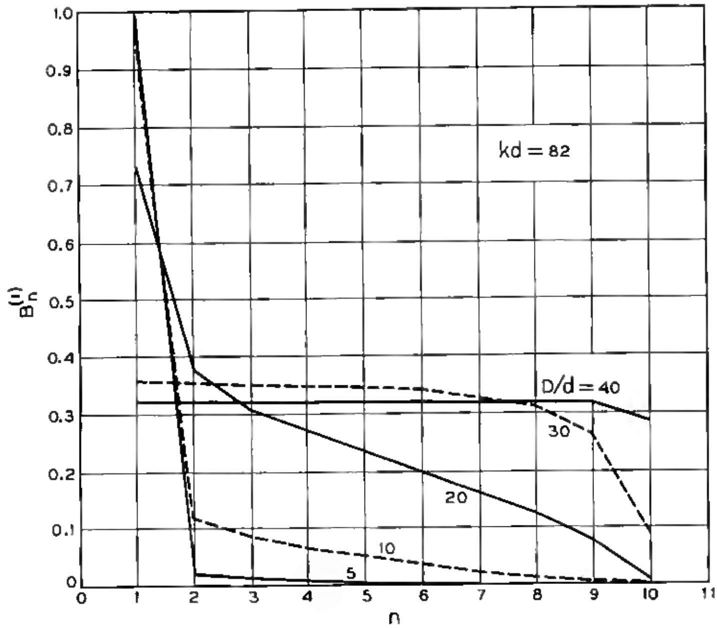

Fig. 1—Steady state power distributions for different values of the correlation length D. The multimode guide carries 10 modes.

mode numbers add up to unity. The most important aspect of Fig. 1 is the shape of the steady state power distribution for different values of D/d, the ratio of correlation length to slab half width. For very long correlation length each mode carries an equal amount of power regardless of the shape of the power versus mode distribution at the beginning of the guide. As D/d decreases more power is carried by the lower-order modes. For very small values of D/d (less than unity) essentially all the power is contained in the lowest-order mode. Figure 2 presents a similar graph for the case of twenty-one modes. The shape of the steady state distributions is essentially unchanged except that similarly shaped curves carry smaller D/d values showing that the number of modes does not affect the general behavior of the steady state distributions.

The shape of the steady state distributions can be explained as follows. For long correlation length only the high-order guided modes lose power directly to the radiation field while the guided modes couple in such a way that only next neighbors exchange power. It is thus understandable that the power tends to equalize among all the modes. For very short correlation length all guided modes couple directly to

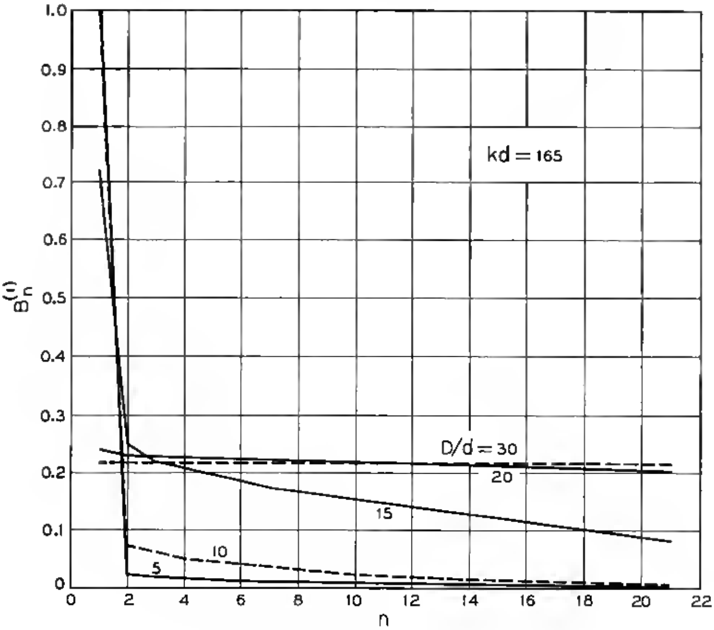

Fig. 2—Same as Fig. 1 for the 21-mode case.

radiation. Higher-order modes lose power by this mechanism at a higher rate than lower-order modes. In addition, each guided mode couples to all the other guided modes. Since the lowest-order mode loses the least power to radiation it is the one that "survives" after all the other modes have lost nearly all of their power. In general, the correlation length of the random core-cladding interface irregularities cannot be chosen at will. However, for multimode operation one would hope for a long correlation length which makes it possible to transmit power in all the modes. Coupling with short correlation length forces the multimode fiber into single-mode steady state operation.

Figure 3 shows the normalized steady state loss $\alpha d/(\bar{\sigma}^2 k^2)$ of the slab waveguide (the lines labeled $i = 1$) as functions of D/d. The lines labeled $i = 2$ represent the second eigenvalue of the eigenvalue problem. The important feature of Fig. 3 is the existence of a maximum as a function of D/d and the separation between the curves of the first ($i = 1$) and second ($i = 2$) eigenvalues. With the help of these two curves it is possible to estimate the region where steady state operation has been achieved. The loss parameters $\alpha^{(1)}$ and $\alpha^{(2)}$ enter in the form $\exp(-\alpha^{(i)}z)$ as the first and second term of a series expansion [see

equation (62)]. When $\alpha^{(2)}z > 4.6$ we have $\exp(-\alpha^{(2)}z) < 10^{-2}$ so that the second term of the series expansion is becoming insignificant and steady state is essentially achieved.

Figures 4, 5, and 6 show the way in which an initially uniform distribution of power settles down toward the steady state distribution. Three different values of correlation length were used. $D/d = 0.01$ is a sufficiently small value whose steady state distribution consists of only the lowest-order mode. The second mode carries only 10^{-4} of the power of the first mode at $z \to \infty$. The value $D/d = 20$ was chosen as an example for an intermediate correlation length. The steady state distribution in this case does not favor exclusively the lowest-order mode but assumes a shape in which higher-order modes carry decreasingly smaller amounts of power. The value of $D/d = 35$ is sufficiently large to produce an essentially uniform steady state distribution.

The next three figures, Figs. 7, 8, and 9, show how the steady state distribution establishes itself if initially all the power is launched in the first mode. The last three figures, Figs. 10, 11, and 12, show similar

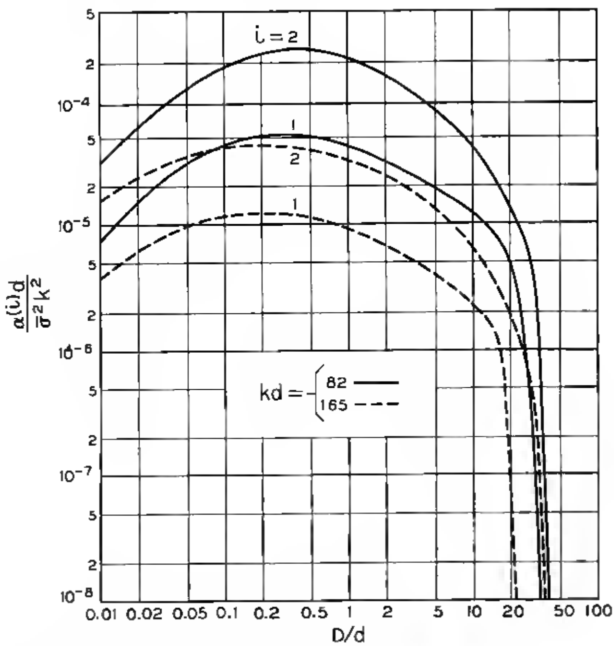

Fig. 3—The normalized first two eigenvalues $\alpha^{(i)}d/(\bar{\sigma}^2k^2)$ ($i = 1$ and $i = 2$) are shown as functions of D/d for the 10- and 21-mode case. The lowest-order eigenvalue, $i = 1$, is the steady state power loss of the multimode waveguide.

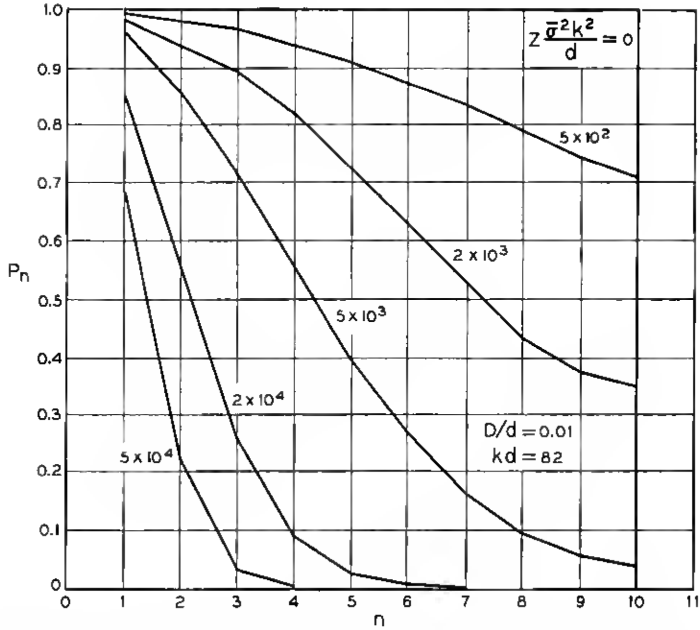

Fig. 4—Power distribution versus mode number for several values of normalized length along the guide for $D/d = 0.01$.

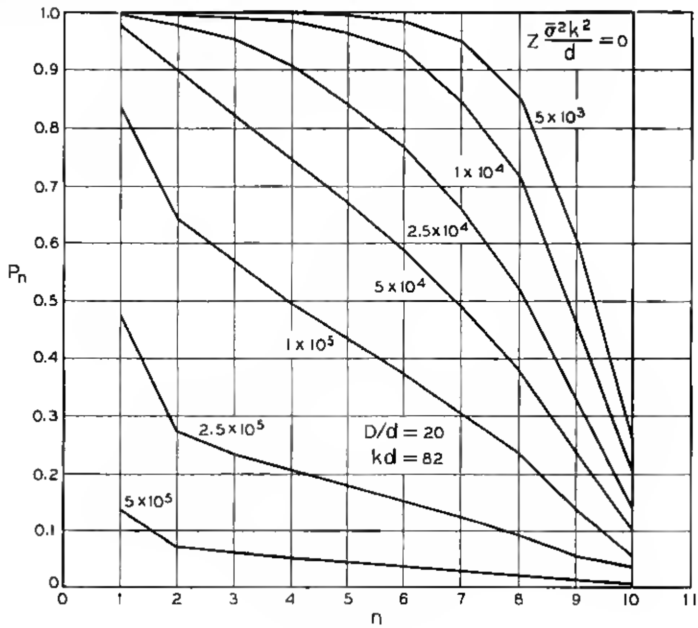

Fig. 5—Same as Fig. 4, $D/d = 20$.

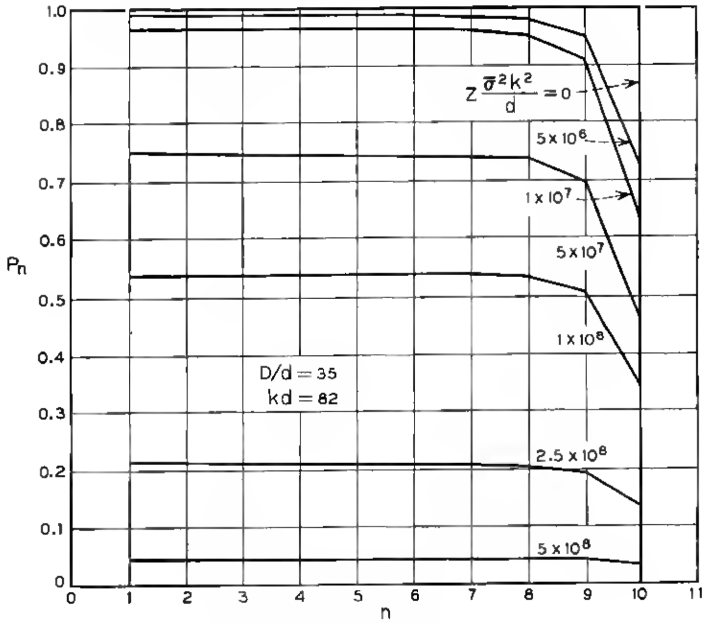

Fig. 6—Same as Fig. 4, $D/d = 35$.

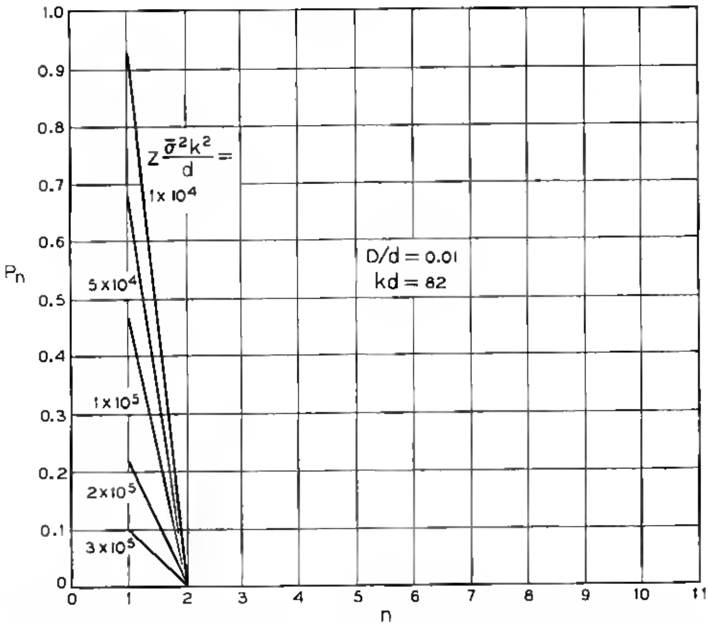

Fig. 7—Power distribution versus mode number for several values of the normalized length along the guide. Only mode 1 is excited at $z = 0$. $D/d = 0.01$.

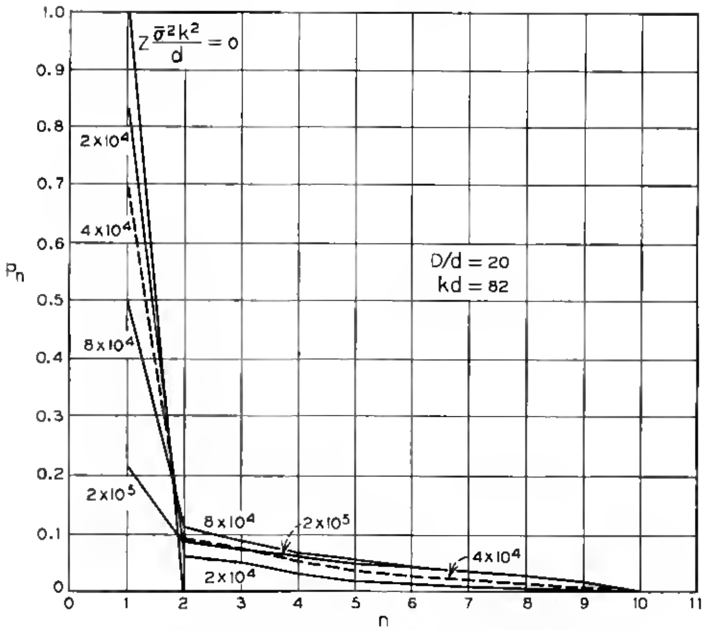

Fig. 8—Same as Fig. 7, $D/d = 20$.

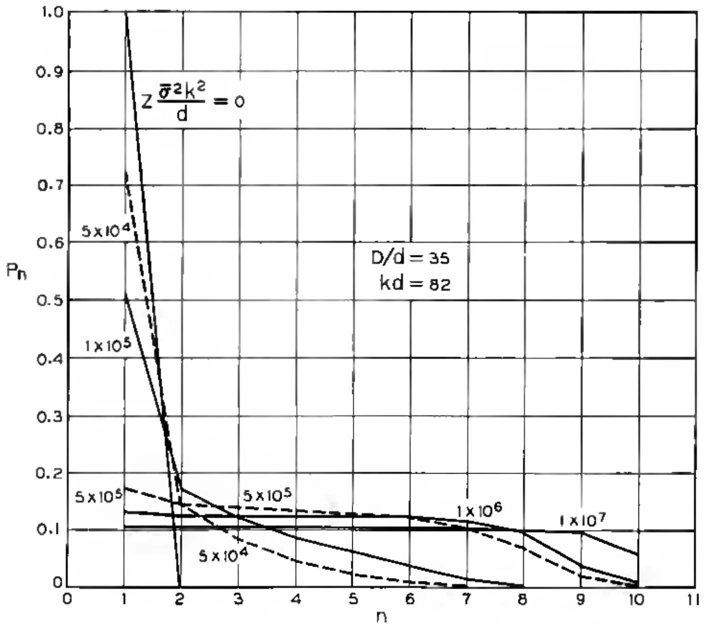

Fig. 9—Same as Fig. 7, $D/d = 35$.

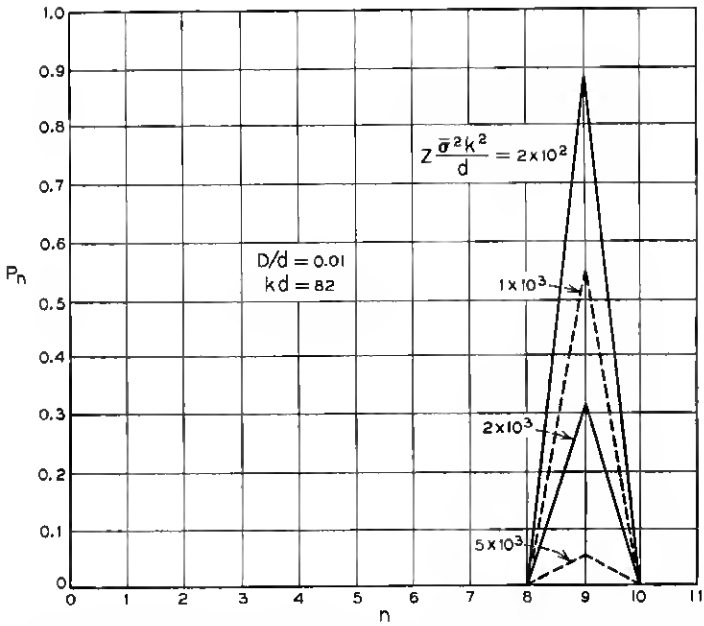

Fig. 10—Power distribution versus mode number for several values of the normalized distance along the guide. Only mode 9 is excited at $z = 0$. $D/d = 0.01$.

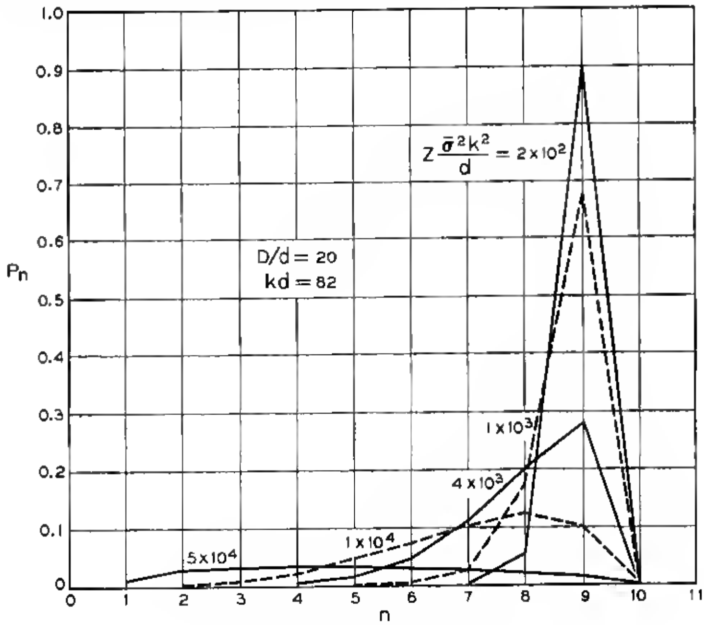

Fig. 11—Same as Fig. 10, $D/d = 20$.

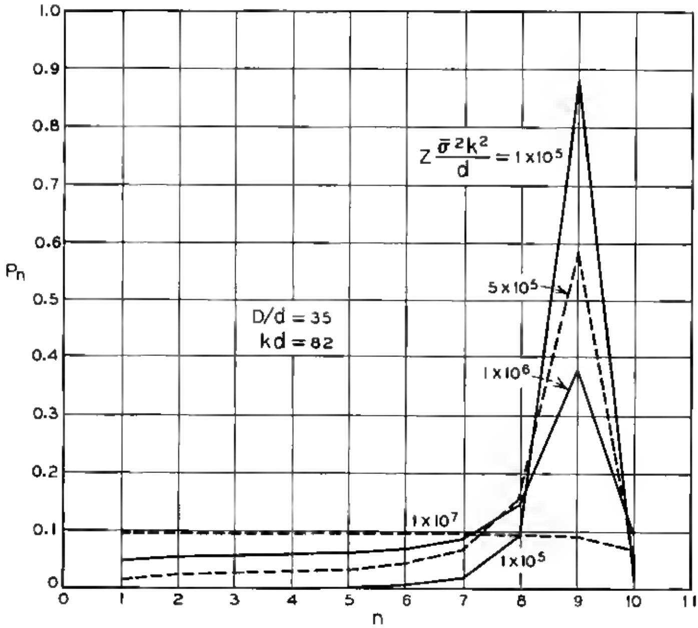

Fig. 12—Same as Fig. 10, $D/d = 35$.

plots for the case that all the power starts out in mode 9. The three values, $D/d = 0.01$, 20, and 35, have again been used.

It may be of interest to know the ratio of the power remaining in the steady state if initially only mode 1 or mode 9 were excited. We call this ratio $P(1, z)/P(9, z)$ and obtain $P(1, z)/P(9, z) = 1.55 \times 10^4$ for $D/d = 0.01$, $P(1, z)/P(9, z) = 9.75$ for $D/d = 20$, and $P(1, z)/P(9, z) = 1.08$ for $D/d = 35$.

Tables I, II, and III show the ten eigenvalues of the steady state mode distributions together with the first eigenvector for the same three values of D/d. The lowest-order and the second eigenvalue appear also in Fig. 3. The other eigenvalues are given in the tables for the sake of completeness. It should be noted that the integer values in the left-hand column of these tables have different meaning for the eigenvalues and the eigenvector. The eigenvalues $\alpha^{(i)}$ are ordered in increasing value and originate as the ten solutions of the eigenvalue problem [equation (60)] of a symmetric 10 by 10 matrix. The first eigenvector $B_\nu^{(1)}$ belongs to the lowest eigenvalue $\alpha^{(1)}$. The subscript ν is a mode label in this case. The eigenvector $B_\nu^{(1)}$ is proportional to the steady state power distribution. Table I shows clearly that mode 1 carries the overwhelming

TABLE I—EIGENVALUES AND THE FIRST EIGENVECTOR FOR $D/d = 0.01$

i or ν	$\frac{d}{\sigma^2 k^2} \alpha^{(i)}$	$B_\nu^{(1)}$
1	7.624×10^{-6}	9.999×10^{-1}
2	3.044×10^{-5}	8.486×10^{-5}
3	6.845×10^{-5}	7.160×10^{-5}
4	1.214×10^{-4}	6.789×10^{-5}
5	1.891×10^{-4}	6.630×10^{-5}
6	2.711×10^{-4}	6.546×10^{-5}
7	3.666×10^{-4}	6.497×10^{-5}
8	4.743×10^{-4}	6.466×10^{-5}
9	5.909×10^{-4}	6.444×10^{-5}
10	7.049×10^{-4}	6.424×10^{-5}

TABLE II—EIGENVALUES AND THE FIRST EIGENVECTOR FOR $D/d = 20$

i or ν	$\frac{d}{\sigma^2 k^2} \alpha^{(i)}$	$B_\nu^{(1)}$
1	5.175×10^{-6}	7.284×10^{-1}
2	1.402×10^{-5}	3.799×10^{-1}
3	4.576×10^{-5}	3.212×10^{-1}
4	1.008×10^{-4}	2.770×10^{-1}
5	1.951×10^{-4}	2.376×10^{-1}
6	3.329×10^{-4}	2.011×10^{-1}
7	5.217×10^{-4}	1.648×10^{-1}
8	7.596×10^{-4}	1.248×10^{-1}
9	1.046×10^{-3}	7.475×10^{-2}
10	4.345×10^{-3}	3.129×10^{-3}

TABLE III—EIGENVALUES AND THE FIRST EIGENVECTOR FOR $D/d = 35$

i or ν	$\frac{d}{\sigma^2 k^2} \alpha^{(i)}$	$B_\nu^{(1)}$
1	6.415×10^{-9}	3.296×10^{-1}
2	2.552×10^{-7}	3.294×10^{-1}
3	1.112×10^{-6}	3.293×10^{-1}
4	4.251×10^{-6}	3.291×10^{-1}
5	7.789×10^{-6}	3.290×10^{-1}
6	1.608×10^{-5}	3.287×10^{-1}
7	3.149×10^{-5}	3.279×10^{-1}
8	5.837×10^{-5}	3.248×10^{-1}
9	1.036×10^{-4}	3.094×10^{-1}
10	1.765×10^{-4}	2.067×10^{-1}

amount of power in the steady state if $D/d = 0.01$. The distribution of the first eigenvector for $D/d = 20$ appears also in Fig. 1.

In order to obtain a feeling for the amount of irregularity of the core-cladding interface that will cause a certain amount of loss, we consider the case $kd = 82$ (10 modes). For $\lambda = 1\ \mu m$ we then obtain $d = 13\ \mu m$ for the slab half width. We now ask the question: What value of the rms deviation $\bar{\sigma}$ causes 10 dB/km steady state radiation loss? The result is obtained from Fig. 3 or from Tables I, II, and III and is shown in Table IV. The tolerance requirements, arising from the need

TABLE IV—RMS DEVIATION OF CORE-CLADDING INTERFACE CAUSING 10 dB/km LOSS FOR $kd = 82$, $\lambda = 1\ \mu m$

D/d	$\bar{\sigma}/d$	$\bar{\sigma}(\mu m)$
0.01	7.65×10^{-4}	9.94×10^{-3}
0.3	3.26×10^{-4}	4.25×10^{-3}
20.0	9.30×10^{-4}	1.21×10^{-2}
35.0	2.63×10^{-2}	3.42×10^{-1}

for keeping the steady state radiation losses low, are thus very stringent since the rms deviation of the core-cladding interface must be kept within fractions of micrometers.

III. APPLICATION TO DELAY DISTORTION*

Multimode waveguides suffer from delay distortion that occurs because the modes contributing to the power transmission travel with different group velocities. Modes with a higher group velocity arrive at the receiver earlier than modes with a slower group velocity. A pulse, whose power is shared in some way by many modes, is thus distorted and lengthened by this effect. If the modes exchange power rapidly among each other this pulse lengthening effect of multimode waveguides can be substantially reduced. S. D. Personick[4] first pointed out the beneficial effect of tight mode coupling for the reduction of pulse delay distortion. For a two-mode waveguide Personick's results have been confirmed by a rigorous analysis by H. E. Rowe and D. T. Young.[5] Our present work has some applications to the reduction of delay distortion by mode mixing. It is clear that if the coupling between the modes is strong, as would be desirable for delay distortion reduction,

* A more rigorous discussion of pulse distortion in multimode waveguides will be published in a later issue of B.S.T.J.[3]

the steady state power distribution is reached sooner. From Fig. 1 we see that only if the correlation length is large do many modes contribute to energy transport in the waveguide. As far as delay distortion is concerned it might appear advantageous to operate with a short correlation length forcing the multimode waveguide into essentially single-mode steady state operation. However, this method has the disadvantage that most of the power that is initially launched into higher-order modes is lost by radiation so that the waveguide suffers high transient losses. If a light emitting diode is to be used as the transmitter, single-mode operation is most undesirable. That leaves us only with the choice of a long correlation length (if indeed we have a choice) to reduce the power loss from high-order modes. In the limit of very long correlation length all the modes are excited equally strongly in the steady state distribution. If we can still provide strong coupling between the modes there is a chance that the power will be exchanged among all the modes making possible the reduction of delay distortion by mode mixing. Mode mixing takes place via coupling between nearest neighbors in case of long correlation length. The diffusion of power from mode 1 to the highest-order mode and vice versa is thus likely to be slow.

We can get a rough idea of the "speed" with which the power travels from mode 1 to mode 9 (or from mode 9 to mode 1) from Figs. 9 and 12. It is apparent from both figures that it takes approximately $z\bar{\sigma}^2\mathrm{k}^2/d = 10^{-6}$ to 10^{-7} before the mode at the other end of the mode spectrum has received an appreciable amount of power from the mode that is initially excited. The same power diffusion must, of course, take place for any other excitation of the modes. But the effect becomes observable when we launch all the power in one mode and watch how it redistributes itself over the other modes. This redistribution of power is part of the transient behavior that results in the steady state distribution. It is thus possible to estimate the distance that is required for one transit of power from mode 1 to mode 9 (or from mode 9 to mode 1) by looking at the second eigenvalue. We know that the steady state is reached as soon as the second term in the series expansion (of power in terms of steady states) becomes negligible compared to the leading first term [see equation (62)]. The second term is quite small when $\alpha^{(2)}z = 2.3$. We thus define a diffusion length L_d by the relation

$$L_d = \frac{2.3}{\alpha^{(2)}}. \tag{1}$$

L_d is the distance along the waveguide that is required for the power in one of the modes at the end of the mode spectrum to transfer an appre-

ciable amount of power (which is a somewhat undefined quantity) across to the mode at the other end of the mode spectrum. For $D/d = 35$ we obtain from Table III and equation (1) $L_d\bar{\sigma}^2 k^2/d = 9 \times 10^6$. From Figs. 9 and 12 we see that this is indeed a reasonable estimate for the distance required for a power exchange between mode 1 and 9. The definition (1) allows us immediately to determine the steady state power loss that accompanies this power diffusion among the guided modes. The steady state power loss that occurs over a distance $z = L_d$ is given by

$$\alpha^{(1)} L_d = 2.3 \frac{\alpha^{(1)}}{\alpha^{(2)}}. \tag{2}$$

Table V shows a number of values for $\alpha^{(1)} L_d$ for various correlation lengths. The penalty in radiation loss that must be paid for this mode mixing process is relatively high but it improves with increasing correlation length. Mode mixing via the next neighbor power exchange is not likely to be very effective in reducing delay distortion since only a small fraction of power traveling initially in one mode is transferred to the mode at the other end of the mode spectrum in the distance L_d. One might expect that many such diffusion distances would have to fit into the overall length of the guide before delay distortion reduction by mode mixing becomes appreciable. Table V shows that a large correlation length to slab half width ratio is required in order to keep the loss per distance L_d small. Also shown in the table is the normalized exchange length L_d. The numbers were computed for $\mathrm{k}d = 82$, the ten-mode case.

For delay distortion equalization it appears desirable to make L_d much shorter than the total guide length L. If we choose $L/L_d = 100$, for example, we compute from the last column of Table V for $D/d = 40$ with $L = 1$ km, $\mathrm{k}d = 82$, $\lambda = 1\ \mu$m, and $d = 13\ \mu$m the value $\bar{\sigma} = 3.14$ μm for the required rms deviation of the core-cladding interface irregularities. This value is much larger than accidental irregularities need

TABLE V—LOSS PENALTY $\alpha^{(1)} L_d$ AND NORMALIZED POWER EXCHANGE LENGTH L_d FOR $\mathrm{k}d = 82$

D/d	$\alpha^{(1)} L_d$(dB)	$\frac{\bar{\sigma}^2 \mathrm{k}^2}{d} L_d$
20	3.7	1.64×10^5
30	0.33	4.41×10^5
35	0.25	9.02×10^6
40	0.064	3.01×10^8

to be. It is thus conceivable that an optical fiber could be designed with an intentional core-cladding interface irregularity with long correlation length for the purpose of reducing pulse delay distortion.

IV. THE COUPLING COEFFICIENTS

In Ref. 1 coupled power equations were derived from the coupled wave equations. The coupled wave equations have the form

$$\frac{dA_\nu}{dz} = \sum_{\mu \neq \nu}^{N} c_{\nu\mu} A_\mu e^{i(\beta_\nu - \beta_\mu)z}. \tag{3}$$

With the coupling coefficient written as

$$c_{\nu\mu} = K_{\nu\mu} f(z) \cdot \tag{4}$$

and with the assumption that the correlation function of $f(z)$ is Gaussian,

$$\langle f(z) f(z-u) \rangle = \bar{\sigma}^2 e^{-(u/D)^2} \tag{5}$$

($\langle \; \rangle$ indicates an ensemble average), the coupled equations for the average power assume the form

$$\frac{dP_\nu}{dz} = -\alpha_\nu P_\nu + \sqrt{\pi}\, \bar{\sigma}^2 D \sum_{\mu=1}^{N} | K_{\nu\mu} |^2 e^{-[(D/2)(\beta_\nu - \beta_\mu)]^2} (P_\mu - P_\nu). \tag{6}$$

The term $-\alpha_\nu P_\nu$ was added to account for the radiation losses of the modes. Coupling coefficients describing the coupling between the guided modes of a slab waveguide caused by core-cladding interface irregularities were derived in Ref. 2. To obtain the coupling coefficient $c_{\nu\mu}$ from our earlier work, we observe that equations (53) and (60) of Ref. 2 correspond to a solution by perturbation theory of equation (3) for the special case that only the lowest-order even guided TE mode of the slab waveguide is excited. Comparison between the corresponding perturbation solution of (3) and equations (53) and (60) of Ref. 2 allows us to find

$$c_{\nu\mu} = \frac{(n_1^2 - n_2^2)\mathrm{k}^2 a_\nu a_\mu (\gamma_\nu \gamma_\mu)^{\frac{1}{2}}}{2i[| \beta_\nu \beta_\mu | (1 + \gamma_\nu d)(1 + \gamma_\mu d)]^{\frac{1}{2}}} [f(z) - (-1)^{\nu+\mu} h(z)]. \tag{7}$$

The symbols appearing in (7) have the following meaning:

d = core half thickness
n_1 = index of refraction of core material
n_2 = index of refraction of cladding material
$\mathrm{k} = 2\pi/\lambda$ = free-space propagation constant
β_ν = propagation constant of mode ν

$$\gamma_\nu = (\beta_\nu^2 - n_2^2 k^2)^{\frac{1}{2}} \tag{8}$$

$$\kappa_\nu = (n_1^2 k^2 - \beta_\nu^2)^{\frac{1}{2}} \tag{9}$$

$$a_\nu = \begin{cases} \cos \kappa_\nu d & \text{for} \quad \nu = 0, 2, 4 \cdots \\ \sin \kappa_\nu d & \text{for} \quad \nu = 1, 3, 5 \end{cases} \tag{10}$$

$f(z)$ = distortion function of upper core-cladding interface ($f(z) = 0$ indicates a perfect interface at $x = d$)

$h(z)$ = distortion function of lower core-cladding interface ($h(z) = 0$ indicates a perfect interface at $x = -d$).

The propagation constants of the even and odd guided TE modes are obtained with the help of (8) and (9) from the eigenvalue equations. We have for even modes

$$\tan \kappa_\nu d = \frac{\gamma_\nu}{\kappa_\nu} \qquad \nu = 0, 2, 4, \cdots \tag{11}$$

and for odd modes

$$\tan \kappa_\nu d = -\frac{\kappa_\nu}{\gamma_\nu} \qquad \nu = 1, 3, 5, \cdots . \tag{12}$$

With the help of the eigenvalue equations, we can express (10) in the following form:

$$a_\nu = \begin{Bmatrix} (-1)^{\nu/2} \\ (-1)^{(\nu-1)/2} \end{Bmatrix} \frac{\kappa_\nu}{(n_1^2 - n_2^2)^{\frac{1}{2}} k} \quad \begin{matrix} \text{for} \quad \nu = 0, 2, 4 \cdots \\ \text{for} \quad \nu = 1, 3, 5 \cdots . \end{matrix} \tag{13}$$

It is convenient to describe the guided modes in terms of a mode angle θ_ν. We can introduce this angle by the equations

$$\kappa_\nu = n_1 k \sin \theta_\nu , \tag{14}$$

$$\beta_\nu = n_1 k \cos \theta_\nu . \tag{15}$$

Equations (14) and (15) represent the transverse and longitudinal components of the propagation vector of a plane wave in the core and are clearly compatible with (9). The guided mode can be represented as a superposition of two plane waves traveling inside of the core of the slab waveguide. $\pm\theta_\nu$ is the angle that these plane waves form with the waveguide axis.

The function $f(z)$ appearing in (4) is replaced with the sum of $f(z)$ and $h(z)$ in (7). Assuming that the two functions are uncorrelated, and assuming further that they have the same correlation function, we find

$$\langle [f(z) - (-1)^{\nu+\mu} h(z)][f(z-u) - (-1)^{\nu+\mu} h(z-u)] \rangle = 2\bar{\sigma}^2 e^{-(u/D)^2}. \tag{16}$$

By collecting all our results we can finally express the absolute square value of $K_{\nu\mu}$ defined by (4) in the form

$$| K_{\nu\mu} |^2 = \frac{n_1^2 k^2 \sin^2 \theta_\nu \sin^2 \theta_\mu}{2d^2\left(1 + \frac{1}{\gamma_\nu d}\right)\left(1 + \frac{1}{\gamma_\mu d}\right) \cos \theta_\nu \cos \theta_\mu}. \tag{17}$$

For small values of $(n_1/n_2 - 1)$ we have $\theta_\nu \ll 1$. For modes far from cutoff we have, in addition, $\gamma_\nu d \gg 1$. Under these conditions, (17) simplifies to the expression

$$| K_{\nu\mu} |^2 = \frac{n_1^2 k^2 \theta_\nu^2 \theta_\mu^2}{2d^2}. \tag{18}$$

Far from cutoff we can approximate the solution of the eigenvalue equations (11) and (12) as follows:

$$\kappa_\nu d = (\nu + 1)\frac{\pi}{2} \quad \text{for} \quad \nu = 0, 1, 2, 3, 4 \cdots . \tag{19}$$

The mode angles can then be expressed as

$$\sin \theta_\nu = \frac{\pi(\nu + 1)}{2n_1 kd}. \tag{20}$$

Finally, it is important to know the largest mode angle that can occur. Mode guidance ceases to exist when the angle, that the plane-wave components of the guided mode form with the core-cladding interface, exceeds the total internal reflection angle θ_c defined by

$$n_1 \cos \theta_c = n_2 . \tag{21}$$

For $(n_1/n_2 - 1) \ll 1$ we obtain approximately

$$\theta_c = \left[2\left(1 - \frac{n_2}{n_1}\right)\right]^{\frac{1}{2}}. \tag{22}$$

Combining (20) and (22) allows us to find an approximate value for the number of modes N that the slab waveguide can support:

$$N = \frac{2}{\pi} n_1 kd\left[2\left(1 - \frac{n_2}{n_1}\right)\right]^{\frac{1}{2}} - 1. \tag{23}$$

V. RADIATION LOSSES

In order to be able to evaluate the coupled power equations (6) we need convenient approximations for the radiation losses α_ν of the guided modes.

The radiation loss problem has been solved in a general way in Ref. 6. Equation (14) of Ref. 6 gives the radiation losses of the even and odd guide modes.

$$\alpha_\nu = \int_{-n_2 k}^{n_2 k} \langle | F(\beta_\nu - \beta) |^2 \rangle I_\nu(\beta) \, d\beta \tag{24}$$

with

$$I_\nu(\beta) = \frac{(n_1^2 - n_2^2)k^3 n_1 \sin^2 \theta_\nu}{2\pi d \cos \theta_\nu \left(1 + \dfrac{1}{\gamma_\nu d}\right)} \cdot \left[\frac{\rho \cos^2 \sigma d}{\rho^2 \cos^2 \sigma d + \sigma^2 \sin^2 \sigma d} + \frac{\rho \sin^2 \sigma d}{\rho^2 \sin^2 \sigma d + \sigma^2 \cos^2 \sigma d}\right] \tag{25}$$

and with the Fourier coefficient of the core-cladding interface function

$$F(\beta_\nu - \beta) = \frac{1}{\sqrt{L}} \int_0^L f(z) e^{-i(\beta_\nu - \beta)z} \, dz. \tag{26}$$

Equations (10), (13), (14), and (15) have been used to express (25) in this form. In addition, (25) has been multiplied with a factor 2 to account for the fact that both core-cladding interfaces have irregularities (contrary to the assumption in Ref. 6) that are statistically independent of each other but have the same correlation function. There are two new parameters in (25):

$$\rho = (n_2^2 k^2 - \beta^2)^{\frac{1}{2}} \tag{27}$$

and

$$\sigma = (n_1^2 k^2 - \beta^2)^{\frac{1}{2}}. \tag{28}$$

The parameter β is the propagation constant (in z direction) of the radiation modes. Using (5) and assuming that $L \gg D$ we obtain from (26)

$$\langle | F(\beta_\nu - \beta) |^2 \rangle = \sqrt{\pi} \, \bar{\sigma}^2 D e^{-[(D/2)(\beta_\nu - \beta)]^2}. \tag{29}$$

The loss expression (24) must be simplified before it can be used for our purposes. We are interested only in multimode waveguides with $kd \gg 1$. The functions $\sin \sigma d$ and $\cos \sigma d$ thus vary rapidly as functions of β. It is impossible to obtain an approximation for all values of D/d. We begin by assuming $D/d \ll 1$. In this case, we can replace the exponential function in (29) by unity and obtain

$$\alpha_\nu = \frac{(n_1^2 - n_2^2)k^3 n_1 \sin^2 \theta_\nu}{2\sqrt{\pi}\cos\theta_\nu\left(1 + \frac{1}{\gamma_\nu d}\right)} \bar{\sigma}^2 \frac{D}{d}$$

$$\cdot \int_{-n_2 k}^{n_2 k} \left[\frac{\rho \cos^2 \sigma d}{\rho^2 \cos^2 \sigma d + \sigma^2 \sin^2 \sigma d} + \frac{\rho \sin^2 \sigma d}{\rho^2 \sin^2 \sigma d + \sigma^2 \cos^2 \sigma d}\right] d\beta. \tag{30}$$

Consider the terms in the integrand. The first term can be written

$$G_1 = \frac{\rho \cos^2 \sigma d}{\rho^2 \cos^2 \sigma d + \sigma^2 \sin^2 \sigma d} = \frac{\rho \cos^2 \sigma d}{\rho^2 + (\sigma^2 - \rho^2)\sin^2 \sigma d}. \tag{31}$$

The sine and cosine functions oscillate rapidly while ρ is only a slowly varying function. The contribution of the second term in the denominator is slight since this term vanishes when the cosine term in the numerator assumes its maximum. On the other hand, when the second term in the denominator is largest, the numerator is zero so that the value of the denominator does not matter. We can thus write to a crude approximation

$$G_1 \approx \frac{\cos^2 \sigma d}{\rho}. \tag{32}$$

The average value of the cosine square function is 1/2 so that we approximate further

$$G_1 \approx \frac{1}{2\rho}. \tag{33}$$

It appears that this approximation may be very poor at $\rho = 0$. However, by converting the integration variable from ρ to β, we see that

$$d\beta = -\frac{\rho}{\beta}\, d\rho \tag{34}$$

showing that there is no pole at $\rho = 0$. By an analogous argument we find that the second term in the integrand can also be approximated as

$$G_2 \approx \frac{1}{2\rho}. \tag{35}$$

The integral in (30) thus assumes the value

$$\int_{-n_2 k}^{n_2 k} (G_1 + G_2)\, d\beta \approx \int_{-n_2 k}^{n_2 k} \frac{1}{\rho}\, d\beta = \pi. \tag{36}$$

For $D/d < (1/2n_2kd)$ we obtain the following approximation for the radiation loss of the νth guided mode of the slab waveguide:

$$\alpha_\nu = \frac{\sqrt{\pi}\,(n_1^2 - n_2^2)n_1 k^3 \sin^2 \theta_\nu}{2 \cos \theta_\nu \left(1 + \frac{1}{\gamma_\nu d}\right)} \bar{\sigma}^2 \frac{D}{d}. \tag{37}$$

Next we try to obtain an approximation for large values of the correlation length, $D/d \gg 1$. The general expression for the radiation loss of the νth guided mode follows from (24), (25), and (29):

$$\alpha_\nu = \frac{(n_1^2 - n_2^2)n_1 k^3 \sin^2 \theta_\nu}{2\sqrt{\pi}\left(1 + \frac{1}{\gamma_\nu d}\right) \cos \theta_\nu} \bar{\sigma}^2 \frac{D}{d}$$
$$\cdot \int_{-n_2 k}^{n_2 k} e^{-[(D/2)(\beta_\nu - \beta)]^2}$$
$$\cdot \left[\frac{\rho \cos^2 \sigma d}{\rho^2 \cos^2 \sigma d + \sigma^2 \sin^2 \sigma d} + \frac{\rho \sin^2 \sigma d}{\rho^2 \sin^2 \sigma d + \sigma^2 \cos^2 \sigma d}\right] d\beta. \tag{38}$$

The exponential function under the integral sign decreases very rapidly with increasing values of $\beta_\nu - \beta$ for $D/d \gg 1$. Since the largest value that β can assume is $\beta = n_2 k$ only the immediate vicinity of the upper limit of the integration range contributes to the integral. In this region we have $\rho \ll n_2 k$. In order to be able to work out approximations for the case of large correlation length we must consider two more subdivisions, the case that D/d is small enough so that the exponential factor under the integral sign in (38) varies slowly compared to the rapid oscillations of the sine and cosine functions and the opposite case where the exponential function decays to insignificant values within one cycle of the oscillations of the oscillatory functions.

In the first case, slowly varying exponential function, we can consider ρ and σ approximately constant over one cycle of oscillation except for the σd term appearing in the argument of the oscillatory functions and consider the average of the integral over one period

$$\frac{1}{\beta_2 - \beta_1} \int_{\beta_1}^{\beta_2} \frac{\rho \cos^2 \sigma d}{\rho^2 \cos^2 \sigma d + \sigma^2 \sin^2 \sigma d} d\beta$$
$$\approx \frac{\rho}{2\pi} \int_0^{2\pi} \frac{\cos^2 x}{\rho^2 \cos^2 x + \sigma^2 \sin^2 x} dx \approx \frac{1}{\sigma + \rho}. \tag{39}$$

And, similarly, for the second term of the integrand we find

$$\frac{1}{\beta_2 - \beta_1} \int_{\beta_1}^{\beta_2} \frac{\rho \sin^2 \sigma d}{\rho^2 \sin^2 \sigma d + \sigma^2 \cos^2 \sigma d} d\beta \approx \frac{1}{\sigma + \rho}. \tag{40}$$

Since only small values of ρ can contribute to the integral, because of the rapid decay of the exponential function, we use the approximations

$$\frac{1}{\sigma + \rho} \approx \frac{1}{\sqrt{n_1^2 - n_2^2\mathrm{k}}} \tag{41}$$

and

$$\beta = n_2\mathrm{k} - \frac{\rho^2}{2n_2\mathrm{k}}. \tag{42}$$

Using all these approximations and the change of integration variable

$$d\beta = -\frac{\rho}{\beta}\,d\rho \approx -\frac{\rho}{n_2\mathrm{k}}\,d\rho, \tag{43}$$

the integral in (38) can be approximated by the following expression:

$$\begin{aligned}\int_{-n_2\mathrm{k}}^{n_2\mathrm{k}} \frac{2}{\sigma + \rho}\, e^{-[(D/2)(\beta_\nu - \beta)]^2}\, d\beta &\approx \frac{2}{n_2\mathrm{k}^2\sqrt{n_1^2 - n_2^2}} \\ &\quad \cdot \int_0^\infty \rho \exp\left\{-\left[\frac{D}{2}\left(\beta_\nu - n_2\mathrm{k} + \frac{\rho^2}{2n_2\mathrm{k}}\right)\right]^2\right\} d\rho \\ &= \frac{4}{\mathrm{k}\sqrt{n_1^2 - n_2^2}\, D} \int_{[(D/2)(\beta_\nu - n_2\mathrm{k})]}^\infty e^{-u^2}\, du \\ &= \frac{2\sqrt{\pi}}{\mathrm{k}D\sqrt{n_1^2 - n_2^2}} \left\{1 - \operatorname{erf}\left[\frac{D}{2}(\beta_\nu - n_2\mathrm{k})\right]\right\}. \end{aligned} \tag{44}$$

We have now finally obtained the result that the radiation loss of the νth mode can be approximated by

$$\alpha_\nu = \frac{n_1\mathrm{k}^2\sqrt{n_1^2 - n_2^2}\sin^2\theta_\nu}{d\left(1 + \frac{1}{\gamma_\nu d}\right)\cos\theta_\nu}\, \bar{\sigma}^2 \left\{1 - \operatorname{erf}\left[\frac{D}{2}(\beta_\nu - n_2\mathrm{k})\right]\right\}. \tag{45}$$

The range of applicability of (45) is obtained by considering tbat we must require the exponential function in (44) to change only sligbtly over the range $\Delta\rho$ corresponding to $\Delta(\sigma d) = 2\pi$. This condition can be expressed as

$$\frac{\rho\Delta\rho D^2}{2n_2\mathrm{k}}\left(\beta_\nu - n_2\mathrm{k} + \frac{\rho^2}{2n_2\mathrm{k}}\right) \ll 1. \tag{46}$$

The increment $\Delta\rho$ is obtained from

$$\Delta\rho = 2\pi\,\frac{\sigma}{\rho}\,\frac{1}{d}. \tag{47}$$

The condition (46) must hold primarily for small values of ρ. We use, therefore,

$$\rho = \eta\, \Delta\rho. \tag{48}$$

With η being a small number such as 2 or 5 and obtained from (46), (47), and (48) with $\sigma \approx (n_1^2 - n_2^2)^{\frac{1}{2}}\mathrm{k}$,

$$\frac{2}{(\beta_\nu - n_2\mathrm{k})d + \dfrac{(n_1^2 - n_2^2)\mathrm{k}d}{2n_2}} < \frac{D}{d} < \left\{\frac{n_2}{\pi\left[(\beta_\nu - n_2\mathrm{k})d + \eta\pi\,\dfrac{(n_1^2 - n_2^2)^{\frac{1}{2}}}{n_2}\right](n_1^2 - n_2^2)^{\frac{1}{2}}}\right\}^{\frac{1}{2}}. \tag{49}$$

The left-hand side of this inequality follows from the requirement that the exponential function in (44) must drop to small values as ρ grows from 0 to approximately $(n_1^2 - n_2^2)^{\frac{1}{2}}\mathrm{k}$.

Finally, we obtain an approximation for very large values of D/d if we assume that the exponential factor in (38) decreases very appreciably over an interval corresponding to one oscillation period of the oscillatory functions. We can now use the approximation

$$\sigma = (n_1^2 - n_2^2)^{\frac{1}{2}}\mathrm{k} \tag{50}$$

treating σ as independent of ρ. Expanding the factor that multiplies the exponential function in (38) in a power series in terms of ρ at $\rho = 0$, keeping only the first none-vanishing term, results in

$$\int_0^\infty \frac{\rho}{\beta} e^{-[(D/2)(\beta_\nu - \beta)]^2} \cdot \left[\frac{\rho\cos^2\sigma d}{\rho^2\cos^2\sigma d + \sigma^2\sin^2\sigma d} + \frac{\rho\sin^2\sigma d}{\rho^2\sin^2\sigma d + \sigma^2\cos^2\sigma d}\right] d\rho$$

$$\approx \frac{1}{n_2(n_1^2 - n_2^2)\mathrm{k}^3}(\cot^2\sigma d + \tan^2\sigma d) \cdot \int_0^\infty \rho^2 \exp\left\{-\left[\frac{D}{2}\left(\beta_\nu - n_2\mathrm{k} + \frac{\rho^2}{2n_2\mathrm{k}}\right)\right]^2\right\} d\rho$$

$$\approx \frac{2(\pi n_2\mathrm{k})^{\frac{1}{2}}}{\mathrm{k}^2(n_1^2 - n_2^2)(\beta_\nu - n_2\mathrm{k})^{\frac{1}{2}}D^3}(\cot^2\sigma d + \tan^2\sigma d)e^{-[(D/2)(\beta_\nu - n_2\mathrm{k})]^2}. \tag{51}$$

If $\beta_\nu = n_2\mathrm{k}$, (51) becomes infinitely large. The approximation leading to the solution of the integral is violated in this case and the result

becomes meaningless. However, this violation of the applicability of the approximate solution of (51) can happen only for the highest-order mode and only if it happens to be directly at its cutoff frequency. Our approximation, if applied to this case, gives a loss value that is too large. Using too large a radiation loss for the highest-order mode affects the power distribution in all the other modes only slightly. The radiation loss of the highest-order mode is large in any case. Power coupled from the neighboring guided modes to this mode is lost rapidly. Using too large a loss value for this mode makes little difference to any of the other modes. We thus use (51) for all values of β_ν.

A more serious violation of the applicability of (51) occurs if either the function cot σd or the function tan σd should become infinite or at least very large. In both of these cases the integral assumes the form

$$\int_0^\infty \frac{1}{\beta} e^{-[(D/2)(\beta_\nu-\beta)]^2}\, d\rho \approx \frac{1}{n_2 k} \int_0^\infty e^{-[(D/2)(\beta_\nu - n_2 k + (\rho^2/2n_2 k))]^2}\, d\rho$$

$$\approx \frac{\pi^{\frac{1}{2}}}{D[n_2 k(\beta_\nu - n_2 k)]^{\frac{1}{2}}} e^{-[(D/2)(\beta_\nu - n_2 k)]^2}. \tag{52}$$

For very large D/d the radiation loss approximation is

$$\alpha_\nu = \bar{\sigma}^2 \frac{n_1 k^3 \sin^2 \theta_\nu e^{-[(D/2)(\beta_\nu - n_2 k)]^2}}{2d[n_2 k(\beta_\nu - n_2 k)]^{\frac{1}{2}}\left(1 + \dfrac{1}{\gamma_\nu d}\right) \cos \theta_\nu}$$

$$\cdot \begin{cases} \dfrac{2n_2}{kD^2(\beta_\nu - n_2 k)} (\cot^2 \sigma d + \tan^2 \sigma d) & \text{for} \quad \tan \sigma d \neq 0 \\ & \text{and} \quad \cot \sigma d \neq 0 \\ (n_1^2 - n_2^2) & \text{for} \quad \tan \sigma d = 0 \\ & \text{or} \quad \cot \sigma d = 0. \end{cases} \tag{53}$$

Equation (53) holds for values of D/d that are much larger than the D/d values in the range indicated in (49).

VI. THE EIGENVALUE PROBLEM

Knowing the coupling coefficients and the radiation losses allows us to determine the power distribution in the multimode dielectric slab waveguide as a function of the distance z along the guide. Introducing the abbreviations

$$h_{\nu\mu} = \sqrt{\pi}\, \bar{\sigma}^2 D \mid K_{\nu\mu} \mid^2 e^{-[(D/2)(\beta_\nu - \beta_\mu)]^2} \tag{54}$$

and

$$b_\nu = \sum_{\mu=1}^{N} h_{\nu\mu} , \tag{55}$$

we can write the coupled power equations (6) in the form

$$\frac{dP_\nu}{dz} = -(\alpha_\nu + b_\nu)P_\nu + \sum_{\mu=1}^{N} h_{\nu\mu}P_\mu . \tag{56}$$

The trial solution

$$P_\nu = B_\nu e^{-\alpha z} \tag{57}$$

converts (56) into an eigenvalue problem

$$\sum_{\mu=1}^{N} [h_{\nu\mu} - (\alpha_\mu + b_\mu - \alpha)\delta_{\nu\mu}]B_\nu = 0. \tag{58}$$

The coefficient matrix of this problem is real and symmetric as can be seen from (54) and the condition (11) of Ref. 1. This latter condition can be expressed in the form

$$| K_{\nu\mu} |^2 = | K_{\mu\nu} |^2. \tag{59}$$

The symmetry condition (59) follows also directly from (17). The eigenvalue α is obtained from the eigenvalue equation

$$| h_{\nu\mu} - (\alpha_\mu + b_\mu - \alpha) \delta_{\nu\mu} | = 0. \tag{60}$$

The vertical lines in (60) indicate that the determinant of the matrix, whose $\nu\mu$ element appears explicitly, must be formed. The eigenvalue equation is an algebraic equation of order N providing N different solutions for the eigenvalues $\alpha^{(i)}$. The eigenvectors, whose elements are $B_\nu^{(i)}$, are mutually orthogonal and will be assumed to be normalized,

$$\sum_{\nu=1}^{N} B_\nu^{(i)}B_\nu^{(j)} = \delta_{ij} . \tag{61}$$

The general solution of (56) can now be expressed as a linear superposition of the N eigensolutions,

$$P_\nu(z) = \sum_{i=1}^{N} c_i B_\nu^{(i)} e^{-\alpha^{(i)} z}. \tag{62}$$

The expansion coefficients c_i must be determined from the given power distribution at $z = 0$. With the help of (61) we obtain from (62)

$$c_i = \sum_{\nu=1}^{N} B_\nu^{(i)} P_\nu(0). \tag{63}$$

VII. CONCLUSIONS

In this paper we have shown that coupling between the guided modes of a multimode waveguide causes the power versus mode number distribution to settle down to a steady state provided the signal is allowed to travel far enough in the waveguide. This steady state applies, of course, only to the CW case. For very long correlation length of the core-cladding interface irregularities the steady state distribution contains equal power in all the modes. For very short correlation length, on the other hand, only the lowest-order mode carries an appreciable amount of power, forcing the fiber into single-mode steady state operation.

The results of this paper have some application to delay distortion equalization. If the power carried by the guided modes is exchanged rapidly among them, the pulse distortion caused by the different group velocities of the modes is partially compensated. Coupling among the modes is of necessity accompanied by radiation losses. Effective pulse delay distortion equalization has a chance of working only if the correlation length of the core-cladding irregularities is long since the penalty paid in radiation loss becomes high for short correlation length. In addition, only for long correlation length do all the modes carry power in the steady state distribution.

A detailed discussion of the numerical results and the properties of multimode waveguides is to be found at the beginning of the paper.

REFERENCES

1. Marcuse, D., "Derivation of Coupled Power Equations," B.S.T.J., *51*, No. 1 (January 1972), pp. 229–237.
2. Marcuse, D., "Mode Conversion Caused by Surface Imperfections of a Dielectric Slab Waveguide," B.S.T.J., *48*, No. 10 (December 1969), pp. 3187–3215.
3. Marcuse, D., "Pulse Propagation in Multimode Dielectric Waveguides," to be published in B.S.T.J.
4. Personick, S. D., "Time Dispersion in Dielectric Waveguides," B.S.T.J., *50*, No. 3 (March 1971), pp. 843–859.
5. Rowe, H. E., and Young, D. T., "Transmission Distortion in Multimode Random Waveguides," to be published in IEEE Trans. MTT.
6. Marcuse, D., "Radiation Losses of Dielectric Waveguides in Terms of the Power Spectrum of the Wall Distortion Function," B.S.T.J., *48*, No. 10 (December 1969), pp. 3233–3242.